How to protect banknotes using moiré fringes

Suggestion of a systematic approach

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July 1975
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Contents

Note

Dr Peter Koeze joined De Nederlandsche Bank, the National Central Bank of the Netherlands, in 1974. This paper is the first he wrote in his new position; it appeared in July 1975. Copies were circulated within the Bank and Messrs. Joh. Enschedé en Zonen, printers to De Nederlandsche Bank. They were also sent, on request, to delegates of other National Central Banks participating in the (standing) Banknote Printers' Conference. A lecture demonstrating the multiple moiré effect and the use of screen traps he gave for an audience drawn from the Bank and Enschedé's on 24 October 1975.

Copies were produced from the original typescript only. This digital version, composed in April 2007, follows the original with minor emendations.

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And all these papers are sealed with the seal of the Great Khan. The procedure of issue is as formal and as authoritative as if they were made of pure gold or silver. On each piece of money several specially appointed officials write their names, each setting his own stamp. When it is completed in due form, the chief of the officials deputed by the Khan dips in cinnabar the seal or bull assigned to him and stamps it on the top of the piece of money so that the shape of the seal in vermilion remains impressed upon it. And then the money is authentic. And if anyone were to forge it, he would suffer the extreme penalty.

The travels of Marco Polo. (Translated by Ronald Latham, The Folio Society, London 1968).

0 Summary

Presently, the most menacing of all possible banknote forgeries is a screened three or four colour reproduction in litho-offset. The moiré effect might provide an effective defence against them. In this paper the moiré effect is treated with special reference to banknotes.

In the first part of the paper a geometrical introduction to the moiré effect is given. The case of two gratings of uniformly spaced straight rulings is dealt with. In the second part a treatment based on symmetry and group theory is presented. The main conclusions of the paper are collected in a separate chapter. They are followed by a general procedure how to design an effective pattern which exhibits in combination with a forger's printing screen conspicuous moiré fringes. It is suggested to design such a pattern not in the two-dimensional space of a banknote but in a transformed space, the spatial frequency domain. The use of a computerized plotter, such as Coragraph, could be of a great advantage. It is proposed to call such a pattern a "screen trap".

Finally, the simplest pattern for the occurrence of moiré fringes with any symmetrical printing screen in any direction is developed. In the last chapter a selection of examples is presented. They prove that a systematic, mathematical treatment of the subject does not limit the scope for creativity, but leaves room for the designer to follow his ideas.

1 Introduction

Of old printers have been haunted by apparitions when reproducing photographs originally produced in autotype. A seemingly innocent autotypical photograph reproduced with an equally seemingly innocent printers' screen may be utterly spoiled by a pattern which seems to origin from nowhere. These apparitions are commonly known as moiré fringes. As so often, this nuisance also may be used to advantage after some thought. If brought about deliberately, it seems to provide an excellent means to turn away intended forgers from autotypical forgeries of banknotes, at the moment the most menacing of all techniques. The traditional means by which banknotes are protected against forgeries (eg, guilloches and photographically inseparable colours) were developed in defence against other reproduction techniques and, consequently, provide a very limited protection against coloured screened reproductions in litho-offset. So, one has to look for additional means to fill this gap in the hedge against intended forgers. A possible solution may be found in the moiré effect, an interference effect well-known in physics. It has been applied already in our Dutch f100-note with some success, as well as in banknotes of other countries. It was felt, however, that the effect could be used more effectively.

The first physicist who seems to have noticed the moiré effect was Lord Rayleigh in 1874. He mentioned it in a paper on the manufacture and theory of diffraction gratings [1]. By large, however, it seems to have escaped the attention of physicists. Although Rayleigh suggested that "this phenomenon might perhaps be made useful as a test" of the accuracy of ruling of diffraction gratings it was not until the fifties that its practicality in metrology was realised. A paper by Oster and Nishijima from 1963 gives a few practical suggestions [2].

Printers, though, were already aware of it at an early time. They tried to avoid the effect as best as they could when reproducing an autotypical photograph. It was on instigation of Dr. O.A. Guinau of Messrs. Joh. Enschedé en Zonen, printers to De Nederlandsche Bank, that Dr. D. Tollenaar, a Dutch chemist, published a mathematical treatment specifically applied to printing in 1945 [3]. Unfortunately, the complete text is available in Dutch only; part of it only has been published in English [4]. It teaches the origin of moiré fringes and how to avoid them. So to speak we have to make a full turn now. We want to bring about moiré fringes deliberately and seek a way to apply them to their full advantage in banknotes in order to scare off intended forgers. We propose to use the term "screen trap" for a design with the desired properties.

This paper intends to provide the necessary theoretical background to designers of banknotes guiding their efforts towards an effective design without seriously limiting their scope. Proving this in chapter 10 some attractive instances done by Messrs. Joh. Enschedé en Zonen are presented.

The paper is divided into two parts. In the first, chapters 2, 3 and 4, a geometrical approach of the moiré effect is given, not very difficult to understand. In the second part, chapters 5, 6, and 7, the subject is treated in a less obvious way. It is based on group theory, a theory often used in physics to describe symmetry effects. Each chapter is followed by a set of conclusions. The attentive reader will notice that often from the theoretician's point of view a conclusion is not sufficiently proved. I think this is acceptable as it is not the purpose of my paper to present a rigid mathematical treatment. It is merely intended to derive a workable set of rules how to apply the moiré effect in banknotes. The conclusions are summarized in chapter 8 to make up a "cooking recipe" and sublimated into a general procedure how to invent an effective design, in which the use of a computerized plotter would come in very handy. The simplest design which generates moiré fringes with any printing screen in any direction is developed in chapter 9. Finally, chapter 10 presents the applications already mentioned.

2 Two parallel gratings of different spacings

If two parallel gratings ruled with uniformly spaced straight lines are superimposed, a pattern of moiré fringes appears. In figure 1 two such gratings are drawn; for easy understanding they are drawn as if not fully overlapping. A pattern of moiré fringes consists of alternately bright and dark fringes. The cause of the phenomenon may be readily understood. If the dark rulings of the gratings coincide, comparatively much light is reflected by the white paper and consequently these places appear bright. On the other hand, places were the dark rulings are neatly arranged along each other, little light is reflected and consequently these places appear dark.

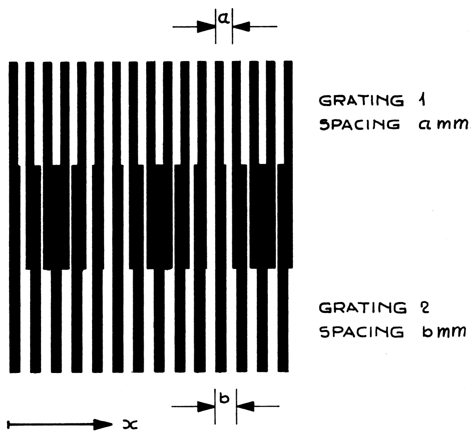

Figure 1. Two parallel gratings of different spacings. For easy understanding the gratings are drawn as if only partly overlapping.

The place co-ordinate x be measured along the normal to both gratings. Assume both gratings to coincide at $x = 0$. As will be seen this assumption does not invalidate the generality of the argument. Now, if the first grating has a spacing of a mm and the second grating a spacing of b mm, then for any x we can always find two numbers m and n such that

$$x = m \cdot a = n \cdot b. \tag{2.1}$$

It is impartial which of the two gratings is called the first and which is called the second, so we may put that

$$a < b \text{ and } m > n. \tag{2.2}$$

The 0-th fringe is found if $m = n = 0$ and $x_0 = 0$, the 1-st fringe is found if $|m_1 - n_1| = 1$ and $x_1 = m_1 a = n_1 b, \ldots$, the p-th fringe is found if $|m_p - n_p| =$

p and $x_p = m_p a = n_p b$. It follows easily that the distance λ between two nearest bright (or dark) moiré fringes, to be called the fringe width, is

$$\lambda = x_{p+1} - x_p = (m_{p+1} - m_p)\, a = (n_{p+1} - n_p)\, b, \tag{2.3}$$

if

$$|m_{p+1} - n_{p+1}| - |m_p - n_p| = p + 1 - p = 1. \tag{2.4}$$

Because $m > n$, it follows

$$m_{p+1} - n_{p+1} - m_p + n_p = 1, \tag{2.5}$$

or

$$m_{p+1} - m_{p+1}\frac{a}{b} - m_p + m_p\frac{a}{b} = 1, \tag{2.6}$$

and thus

$$m_{p+1} - m_p = \frac{b}{b-a}. \tag{2.7}$$

Substitution of (2.7) in (2.3) yields

$$\lambda = \frac{ab}{b-a}. \tag{2.8}$$

Generally, printers prefer to use the number of lines per mm and not the spacing of a grating. It is usual in physics to call this quantity the spatial frequency of the grating expressed in cycles/mm, the dimension being mm^{-1}. So, if we substitute

$$\alpha = \frac{1}{a}\,\mathrm{mm}^{-1} \text{ and } \beta = \frac{1}{b}\,\mathrm{mm}^{-1}, \tag{2.9}$$

it then follows

$$\lambda = \frac{1}{\alpha - \beta}. \tag{2.10}$$

A favourable value of λ might be in the order of 5 mm, in which case

$$\alpha - \beta = 0.2\,\mathrm{mm}^{-1}. \tag{2.11}$$

Conclusion 2.1 *The fringe width of the moiré pattern of two parallel gratings is independent of the absolute value of the spatial frequencies of the gratings; it depends merely on the difference of both.*

Conclusion 2.2 *The spatial frequency of the moiré pattern of two parallel gratings is equal to the difference of the spatial frequencies of the gratings.*

This result is well-known in electronics, where the effect is called “aliasing”.

Conclusion 2.3 *The spatial frequencies of both gratings may not differ too much in order to obtain a suitably large fringe width, eg, a frequency difference of* $0.2\,\mathrm{mm}^{-1}$ *produces a fringe width of* 5 mm.

3 Two oblique gratings of equal spacing

In figure 2 two oblique gratings of equal spacing are drawn. The lines along which both gratings have minimum overlap are indicated by arrows. These

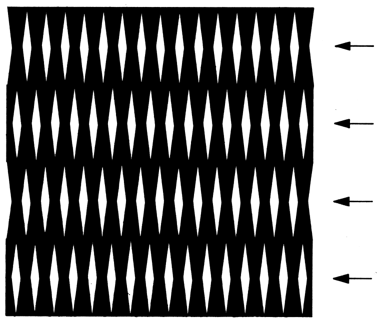

Figure 2. Two oblique gratings of equal spacing. The arrows indicate lines of minimal overlap: bright moiré fringes.

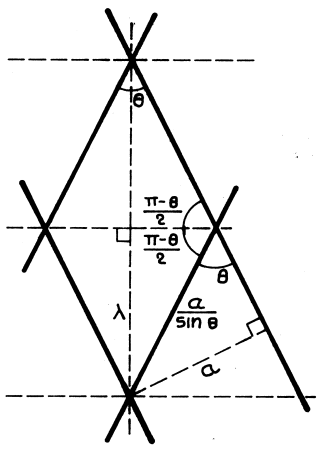

Figure 3. Part of figure 2 enlarged. The spacing of the gratings be a mm, the inter-ruling angle be θ .

lines appear to the human eye by comparison as bright moiré fringes, whereas in between dark moiré fringes are observed. In figure 3 part of the previous figure is shown enlarged with all relevant angles and distances indicated. The spacing of both rulings is again chosen as a mm and the angle between both as θ. The fringe width is also indicated along the perpendicular of the moiré fringes. Now, it follows from simple trigonometric considerations that

$$\lambda = \frac{a}{\sin\theta}\sin\frac{1}{2}(\pi - \theta), \tag{3.1}$$

or

$$\lambda = a\frac{\cos\frac{1}{2}\theta}{\sin\theta}. \tag{3.2}$$

If θ is sufficiently small, this expression may be approximated by

$$\lambda = \frac{a}{\theta}. \tag{3.3}$$

From the same figure it follows that the direction of the fringes bisects the external angle between the rulings of the gratings.

In figure 2 there are other lines which also interconnect the crossing points of the rulings. They are, of course, the other set of diagonals of the rhombus formed by the rulings. It follows again that the fringe width λ' of this set is given by

$$\lambda' = \frac{a}{\sin\theta}\sin\frac{1}{2}\theta. \tag{3.4}$$

The direction of these fringes bisects the internal angle of the rulings of the gratings.

So, we conclude that there are two sets of moiré fringes, perpendicular to each other, one pair being manifest, the other being latent. Which of both is manifest and which is latent depends on the contrasts. The human eye does perceive the set of fringes with the highest contrast between bright and dark fringes. If we look across the figure in a scant direction the paper close to the eye, we can observe the other set of fringes by the apparent change of the fringe width and contrast.

Conclusion 3.1 *The fringe width of the moiré pattern of two oblique gratings with equal frequency depends on the frequency and inter-ruling angle. If the inter-ruling angle is very small, the fringe width is inversely proportional to the angle.*

4 General case: two oblique gratings of different spacings

Now we come to the general case of two oblique gratings of different spacings, as shown in figure 4. Let us call the spacing of the denser grating a mm and the spacing of the less dense grating b mm. If again the spatial frequencies are denoted by Greek letters, then

$$\alpha_0 = \frac{1}{a}\,\mathrm{mm}^{-1} \qquad\qquad \beta_0 = \frac{1}{b}\,\mathrm{mm}^{-1} \tag{4.1}$$

where

$$a < b \text{ and } \alpha > \beta.$$

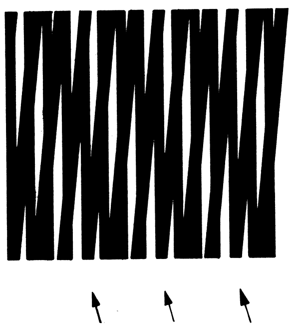

Figure 4. Two oblique gratings of different spacings. The arrows indicate lines of minimal overlap: bright moiré fringes.

In figure 5 the situation is given on a large scale. Both fringe widths are called again λ and λ'. Upon some reflection it follows from trigonometric considerations that twice the area of one half of the rhombus is given by

$$\lambda \cdot p = \lambda' \cdot q = \frac{ab}{\sin\theta}. \tag{4.2}$$

It can be easily deducted that

$$\begin{aligned} p &= \frac{1}{\sin\theta}\left(a^2 + b^2 - 2ab\cos\theta\right)^{\frac{1}{2}} \\ q &= \frac{1}{\sin\theta}\left(a^2 + b^2 + 2ab\cos\theta\right)^{\frac{1}{2}}. \end{aligned} \tag{4.3}$$

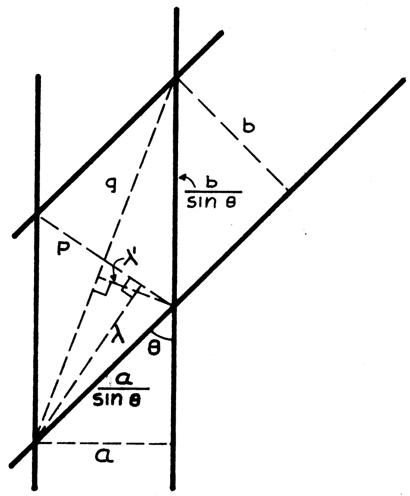

Figure 5. Part of figure 4 enlarged. The spacing of the denser grating be a mm and the other b mm, the inter-ruling angle be θ.

Substituting this in the previous formula yields

$$\lambda = \frac{ab}{(a^2 + b^2 - 2ab\cos\theta)^{\frac{1}{2}}} \tag{4.4}$$

$$\lambda' = \frac{ab}{(a^2 + b^2 + 2ab\cos\theta)^{\frac{1}{2}}}. \tag{4.5}$$

If we put $a = b$ in (4.4) and (4.5), we find after some computation the expressions for λ and λ' derived in the previous chapter, and, if we put $\theta = 0$ in (4.4), we find (2.8), as we ought to of course. Expression (4.4) is maximal for small θ and approximately equal frequencies.

Conclusion 4.1 *The fringe width of the moiré pattern of two oblique gratings with different frequencies depends on the frequencies and the inter-ruling angle. It tends to a maximum for a very small inter-ruling angle and approximately equal frequencies.*

Up to now two attributes of the primary gratings were considered: the frequencies and the inter-ruling angle. There is a third one which is as much of importance. It is the contrast of the gratings between the dark and the bright lines or rather the thickness of the gratings' rulings. They determine the contrast in the moiré pattern. Clearly, if the dark or the bright lines of one grating are very thin, the contrast of the moiré fringes is small. The maximum contrast will be found if dark and bright lines of each grating are of equal thickness.

Conclusion 4.2 *The contrast of the dark and bright moiré fringes in the moiré pattern depends on the thickness of the primary gratings' rulings.*

Conclusion 4.3 *The maximum contrast in the moiré pattern will be found if the dark and bright areas of each primary grating are equal.*

What happens if three gratings of approximately equal frequency are superimposed? As has been demonstrated in (4.4), two of them will produce a moiré pattern with a fringe width considerably larger than the spacing of the third grating. If now the third grating is superimposed, interference of the third grating and the first moiré pattern will produce a second moiré pattern, however with a fairly small fringe width. This means that the first moiré pattern will not be disturbed too much by the third grating.

Conclusion 4.4 *From the viewpoint of security it makes not much sense incorporating easily perceptible moiré fringes as part of a banknote design. The forger's printing screen is not likely to produce large changes of the moiré pattern.*

5 Symmetry of gratings and printing screens

Up to now we have spoken simply of "the" spatial frequency of a grating. In fact, a grating does not have one spatial frequency but a whole range of frequencies. Firstly, the frequency is direction dependent as shown in figure 6. In the direction of the grating's normal ($\varphi = 0$) the spatial frequency is α_0, in another direction the spatial frequency is

$$\alpha_\varphi = \alpha_0 \cos\varphi, \tag{5.1}$$

if φ is the angle with the normal. However, this is not complete yet. Along

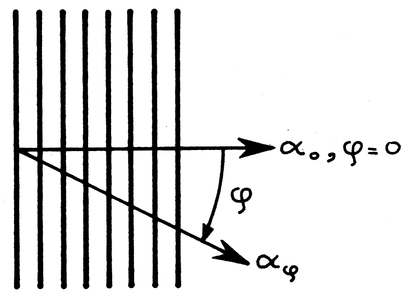

Figure 6. The angular dependency of the spatial frequency of a grating.

the normal ($\varphi = 0$) the frequencies $\alpha_0/2$, $\alpha_0/3$ etcetera are also present. Generally, of the n-th frequency there are n gratings all with a different phase. So, the complete set of frequencies of a grating is given by

$$\begin{aligned} \alpha^1_\varphi &= \alpha_0 \cos\varphi, \\ \alpha^2_\varphi &= \frac{\alpha_0}{2} \cos\varphi, \\ \alpha^3_\varphi &= \frac{\alpha_0}{3} \cos\varphi, \\ \alpha^n_\varphi &= \frac{\alpha_0}{n} \cos\varphi. \end{aligned} \tag{5.2}$$

They may be depicted in a polar diagram as given in figure 7. We could call α^1_φ the principal frequency and α^n_φ the n-th subharmonic.

Conclusion 5.1 *A grating has a set of frequencies, of which the principal frequency is the upper limit.*

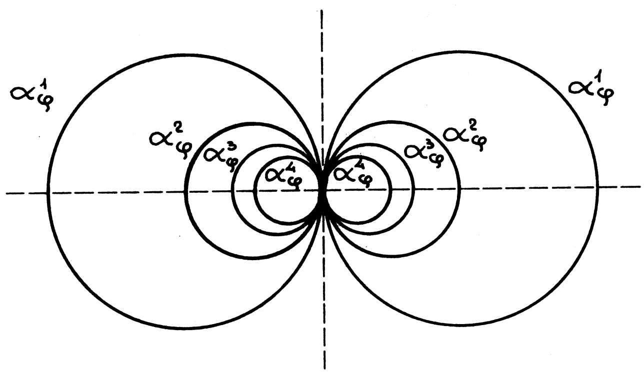

Figure 7. Polar diagram of the spatial frequency of a grating.

Consider again the general case of two oblique gratings with spatial frequencies α_0 and β_0 and θ as the inter-ruling angle, sketched in figure 8. Remember that the bright moiré fringes connect the crossing points of the gratings. Then the following relation must hold in the direction of the moiré fringes

$$\alpha_0 \cos\varphi = \beta_0 \cos(\theta - \varphi)\,, \tag{5.3}$$

if φ is the angle between the moiré fringes and the normal to grating α_0. This relation gives the direction φ of the moiré fringes and, as it is two-valued, of the manifest as well as the latent fringes.

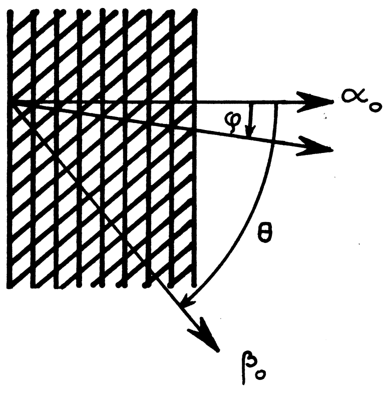

Figure 8. The relationship between α_0, β_0, θ and φ of two oblique gratings.

Conclusion 5.2 *Two gratings will show a moiré pattern in a certain direction if the principal frequency or one of the subharmonics of one grating in that direction is approximately equal to one of the frequencies of the other in that direction (generalisation of conclusion 4.1).*

Thus far we have been considering gratings of uniformly spaced straight lines only. However, moiré fringes may appear with all sorts of symmetrical patterns, eg, printers' halftone screens. They consist of a rectangular array of tiny black squares or dots. They could be considered as being built up of the crossing points of two orthogonal gratings of evenly spaced straight lines. Also in this case, illustrated in figure 9, we can define a spatial frequency, now as the number of dots encountered along a straight line in a direction φ. The frequency here is not a continuous function of φ, but it has a value

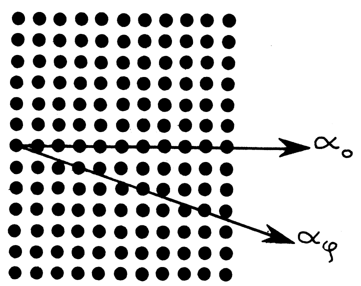

Figure 9. The angular dependency of the spatial frequency of a rectangular array.

for discrete φ only if a line in direction φ goes through a number of points. Analogous to the array which can be thought of as the crossing points of two orthogonal gratings, the polar diagram can be thought of as the crossing points of two orthogonal polar diagrams like figure 7. As it were, the circles of figure 7 are broken up into points as shown in figure 10.

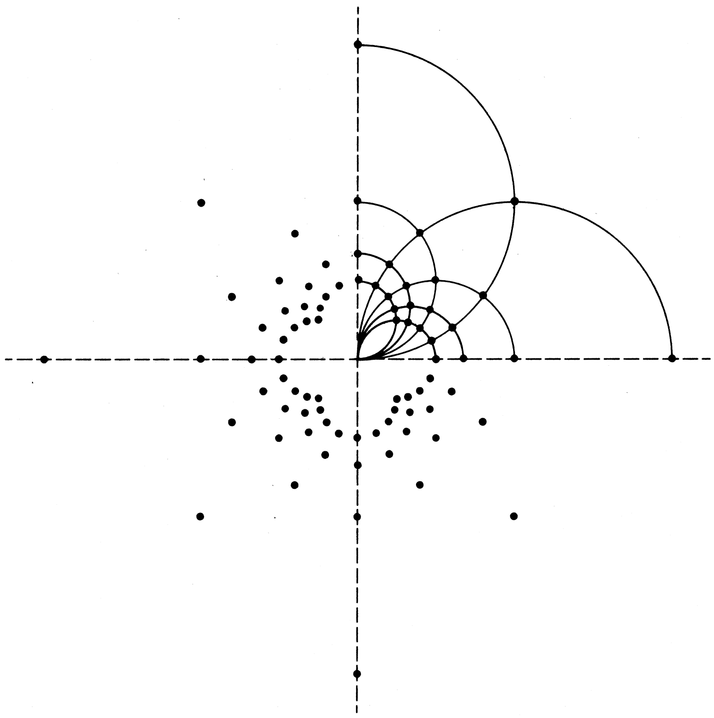

Figure 10. Polar diagram of the spatial frequency of a rectangular array.

6 Group theory of the two-dimensional space

A printing screen and a crystal lattice in physics share an essential feature: they are highly symmetrical. So, it is not too surprising that a theoretical technique used in solid state physics, a branch of physics concerned with crystals, can also be applied fruitfully here. This technique, group theory, is an easy and elegant way to describe symmetry.

What is that, symmetry? Symmetry is the property of geometrical figures to repeat their parts or, more precisely, their property of coinciding with their original position when in different position. Such self-coincidence may be of two types: either the figure shows self-coincidence as a result of a certain movement, or the self-coincidence results from a mirror reflection. It is clear that only those parts which are in some way equal among themselves can be repeated. Unlike objects cannot repeat one another. In the study of symmetry it has long been the accepted practice to consider as being equal not only such figures as may be brought into self-coincidence with one another by simple superposition, but also those which coincide as a result of mirror reflection [5].

The second part of chapter 5 was meant as an early introduction to symmetry. There it was elucidated that a rectangular array of dots can be interpreted as a superposition of two gratings with an inter-ruling angle of $\frac{1}{2}\pi$. Clearly, rotations of the array over π, $\frac{3}{2}\pi$, $-\frac{1}{2}\pi$, $-\pi$, and $-\frac{3}{2}\pi$ produce an array indistinguishable from the original. Such a symmetry is called a 4-fold symmetry, because rotations over multiples of $2\pi/4$ leave the figure unchanged. Of course, other figures can have other rotational symmetries. In general, a pattern has an n-fold symmetry, if rotations over multiples of $2\pi/n$ in the plane of the figure are symmetry operations of the pattern. The centre of rotation is called an n-fold rotation centre. Similarly, a grating of rulings as staged in the first chapters has a 2-fold symmetry because a rotation over π leaves the grating indistinguishable from the original position. Another instance is a corn lattice. It has actually the lowest symmetry possible, a 1-fold symmetry, because it can only be rotated over 2π.

A further symmetry operation which leaves a rectangular array of dots unchanged is a simple translation over the distance between any pair of dots. Clearly, all possible translations of the array are vectorial sums of multiples of the two principal distances.

The two symmetry operations mentioned, rotations and translations, are effected by a movement in the plane of the figure. These are called direct operations. A third symmetry operation called an indirect operation requires a movement out of the plane. It is the mirror reflection, which can be effected by a rotation over π out of the plane about a line in the plane. A mirror reflection followed by a translation along the two-fold rotation axis is called a glide mirror reflection.

Herewith all possible symmetry operations in the two-dimensional space are given: translations, rotations and mirror reflections. All two-dimensional figures one can think of have a symmetry which can be explained as a combination of translations, rotations and mirror reflections. It appears on theoretical grounds, not to be included here, that there is only a countable number of such combinations possible. Each combination is called a group and can be characterized and handled mathematically as a matrix. The total of all groups may be subdivided into classes according to certain common properties.

Some further useful concepts are the order of a group and the fundamental area of a figure. The fundamental area is an area repeated in the figure by the symmetry operations. It may be finite or infinite. The number of elemental symmetry operations of a group is called the order of the group. If the group describing the symmetry of a figure is of high order, it means that many symmetry operations are applicable and that a fundamental area is repeated manifold. An extreme is to be found in a figure with circle symmetry. The centre of a circle is a ∞-fold rotation centre, i.e. in all directions the same pattern is found. The fundamental area is a line radiating from the centre. On the other hand, a low order means that a fundamental area is not repeated so often. An extreme is a completely arbitrary figure like a corn lattice. No part of the lattice is repeated anywhere; every part of the lattice is different. Another example is written text, for which the same holds.

Of course, this is but a short introduction to group theory. There exists a proliferation of books on the subject, of general content or dedicated to a specific subject. A general introduction accompanied by reprints of authoritative papers has been written by Cracknell [6]. Another book by Fejes Tóth on regular figures is of much importance to the present subject [7].

Fejes Tóth distinguishes 37 two-dimensional groups divided in three major classes. The first class of 12 groups contains groups of pure rotation only; they are translation-free. As can be seen very easily the fundamental area must be infinite. They consist necessarily of one n-fold rotation centre and possibly mirror axes through the centre. The second class of 7 groups contains groups of unidirected translations. The fundamental area is again infinite. The third class of 17 groups contains all other groups with at least two non-parallel translations and possibly n-fold rotation centres, mirror and glide mirror axes. It is interesting to note that theoretically 5-fold rotation centres are not possible in any of the groups in this class.

7 The moiré pattern predictable

What is the use of the concepts introduced in the previous chapter? It enables us to describe in exact terms the symmetry of a figure instead of using very broad terms. As an illustration, the symmetry elements of a rectangular array of dots are given in figure 11. This particular constellation of rotation centres and mirror lines is called W_4^1 by Fejes Tóth. Every group has been assigned a similar symbol. If the dots are not round or square but oblong or triangular or other, the symmetry of the array is lower which means that the number of symmetry elements (or order of the group) is less.

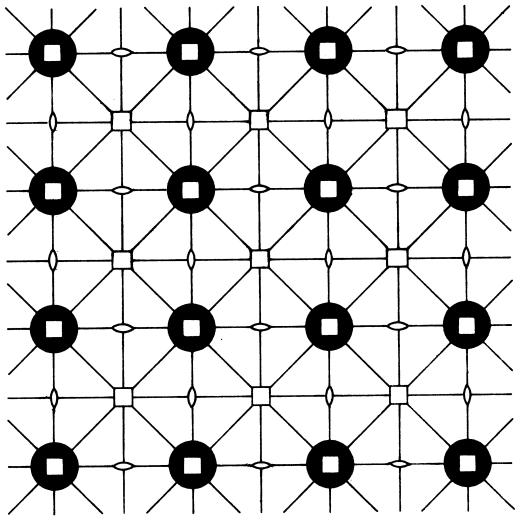

Figure 11. A rectangualar array of black dots and its symmetry elements (group symbol W_4^1).
□ = 4-fold rotation centre
⬯ = 2-fold rotation centre
— = mirror axis

In the case of moiré fringes we have to do with at least two patterns of a certain symmetry: the pattern of the banknote and the forger's halftone printing screen. Given the symmetry of both what will be the symmetry of the moiré pattern? If two patterns are superposed, the symmetry of the total is determined by the common symmetry elements in both. If both patterns have no common elements at all, the resulting symmetry is the lowest possible, a 1-fold rotation centre. This rule may be illustrated with the example of the first chapters. It was shown that the moiré pattern of two gratings of rulings is again a grating of rulings. The symmetry elements of a grating are two different sets of 2-fold centres, two sets of mirror axes, and a translation system (group symbol F_2^1). If two parallel gratings are superposed, the common elements are the mirror axes orthogonal to the rulings and further only those mirror axes of the other set and only those 2-fold rotation centres which are situated at distances given by (2.10) (also group symbol F_2^1). If one of the superposed gratings is slightly rotated with respect to the other, the symmetry is lowered. The mirror axes disappear and only the 2-fold rotation axes remain (group symbol W_2). As was argued before, a corn lattice has the lowest possible symmetry: no mirror axes, no translations, no rotation centres except a trivial 1-fold centre (group symbol E). Therefore a combination of a corn lattice and a pattern of an arbitrary symmetry can produce a moiré pattern of the same trivial symmetry only. Although in reproduction the corn lattice will undoubtedly change, it will do so but in very minor details.

Conclusion 7.1 *The symmetry of a moiré pattern is determined by the common symmetry elements of the primary patterns.*

Conclusion 7.2 *From the viewpoint of security it makes no sense incorporating corn lattices because they will not produce a moiré pattern of a conspicuous nature.*

It may be well to point out that the symmetry of a pattern and the symmetry of its frequency polar diagram are exactly the same as far as mirror axes and rotation centres are concerned. Translations are transformed into frequencies. (In terms of group theory: the point groups of both, subgroups of the complete symmetry groups, are identical.) We have seen this already in the cases of a grating and of a rectangular array of dots. So, instead of using the primary patterns themselves to find the symmetry of the moiré pattern, we may use their polar diagrams as well. It has the added advantage that not only the symmetry of the moiré pattern may be predicted but also the frequencies, that is the whole pattern. If the polar diagrams of the primary patterns are superposed, eg, in the form of transparencies or in the memory of a computer, the common symmetry elements and the common frequencies may be selected easily. They comprise the polar diagram of the moiré pattern. It is then sufficient to construct the inverse transform of this polar diagram to find the moiré pattern. Instead of a trial-and-error method this method may be employed to construct a primary pattern which gives a moiré pattern thought suitable according to certain requirements. Such a procedure is formulated in the next chapter. There it is suggested to call in the help of a computerized plotter such as a Coragraph.

Conclusion 7.3 *The point group of a pattern and of its frequency polar diagram are identical.*

Conclusion 7.4 *It is of advantage to design a screen trap in the spatial frequency domain and not in the two-dimensional space of a banknote.*

Implicitly, it was assumed up to now that patterns were unicoloured. This does not have to be so. If one anticipates the colours a forger is most likely to use, the conspicuity of a moiré pattern in a banknote may be enhanced applying more than one colour chosen to match the forger's colours. This adds a fourth important attribute of a primary pattern to the three already encountered: symmetry, frequency, and optical contrast.

Conclusion 7.5 *The application of well-chosen colours in the screen trap of a banknote may enhance the conspicuity of a moiré pattern.*

8 Directions for use

In the course of reasoning we have come to several conclusions important enough to be set apart. It seems worthwhile to repeat them here and to arrive at some "cooking recipe" how to invent an effective screen trap.

2.1. The fringe width of the moiré pattern of two parallel gratings is independent of the absolute value of the spatial frequencies of the gratings; it depends merely on the difference of both.

2.2. The spatial frequency of the moiré pattern of two parallel gratings is equal to the difference of the spatial frequencies of the gratings.

2.3. The spatial frequencies of both gratings may not differ too much in order to obtain a suitably large fringe width, eg, a frequency difference of $0.2\,\text{mm}^{-1}$ produces a fringe width of 5 mm.

3.1. The fringe width of the moiré pattern of two oblique gratings with equal frequency depends on the frequency and the inter-ruling angle. If the inter-ruling angle is very small, the fringe width is inversely proportional to the angle.

4.1. The fringe width of the moiré pattern of two oblique gratings with different frequencies depends on the frequencies and the inter-ruling angle. It tends to a maximum for a very small inter-ruling angle and approximately equal frequencies.

4.2. The contrast of the dark and bright moiré fringes in the moiré pattern depends on the thickness of the primary gratings' rulings.

4.3. The maximum contrast in the moiré pattern will be found if the dark and bright areas of each primary grating are equal.

4.4. From the viewpoint of security it makes not much sense incorporating easily perceptible moiré fringes as part of a banknote design. The forger's printing screen is not likely to produce large changes of the moiré pattern.

5.1. A grating has a set of frequencies, of which the principal frequency is the upper limit.

5.2. Two gratings will show a moiré pattern in a certain direction if the principal frequency or one of the subharmonics of one grating in that direction is approximately equal to one of the frequencies of the other in that direction (generalisation of conclusion 4.1).

7.1. The symmetry of a moiré pattern is determined by the common symmetry elements of the primary patterns.

7.2. From the viewpoint of security it makes no sense incorporating corn lattices because they will not produce a moiré pattern of a conspicuous nature.

7.3. The point group of a pattern and of its frequency polar diagram are identical.

7.4. It is of advantage to design a screen trap in the spatial frequency domain and not in the two-dimensional space of a banknote.

7.5. The application of well-chosen colours in the screen trap of a banknote may enhance the conspicuity of a moiré pattern.

It is possible now to sublimate the conclusions into a general procedure. It should be remembered that a design is complete only when its four attributes symmetry, frequency, line thickness, and colour are given.

1. Assume one of the two primary patterns not free to choose but given by the circumstances (the forger's pattern).
2. Construct the frequency polar diagram of this primary pattern.
3. Select those symmetry elements and frequencies present in the polar diagram which are to be retained in the moiré pattern desired.
4. Add to these symmetry elements and frequencies any further elements and frequencies one wishes to incorporate in the second primary pattern (the screen trap of the banknote).
5. Transform the resulting polar diagram inversely into the second primary pattern.
6. The second primary pattern has the required, built-in properties.
7. Choose suitable colours.

Of course, we don't know beforehand the forger's printing screen, but we may assume that the forger is likely to use a common commercial halftone screen of 4-fold symmetry. The polar diagram of such a screen has been given in figure 11 on an arbitrary frequency scale. It may be used as the result of step 2 after adjustment of the frequency scale, in which case the actual procedure starts at step 3.

The use of a computerized plotter, such as the Coragraph, could be of great advantage in this procedure. Part of it could very well be automated by means of a computer, particularly the transformation from real space into the spatial frequency domain, the inverse transformation, and the selection of common elements. If a computerized plotter is available, the procedure should be implemented into this machine. Ideally, the designer would do his work at the console of the plotter in a conversational mode.

9 The simplest solution to the problem

The problem we have set ourselves to pursue may be reformulated as follows. Invent a symmetrical pattern which exhibits suitably conspicuous moiré fringes in combination with a rectangular array of dots irrespective of their relative orientation and the frequency of the rectangular array.

So, there are two attributes of our pattern to adjust in the first place: the symmetry and the spatial frequency. In order to fulfil the first requirement of orientation independency, the simplest choice is undoubtedly circle symmetry. Concerning the frequency there is more. We require in fact a pattern in which all frequencies from 0 to ∞ are to be found. Let us call the spatial frequency α again. The radii of the circles which must appear due to the circle symmetry we call r_n counting n from the centre outwards and starting with $n = 0$ in the centre ($r_0 = 0$). Slightly different from before we now define the frequency as half the reciprocal of the circle distance Δr:

$$\alpha\left(r\right) = \frac{1}{2\Delta r} = \frac{1}{2\left(r_n - r_{n-1}\right)}. \tag{9.1}$$

Thus the frequency is not necessarily a constant any more; it may depend on the distance r from the pattern's centre. The simplest choice for the function $\alpha\left(r\right)$ to range from 0 to ∞ will be a linear function of r:

$$\alpha\left(r\right) = kr, \tag{9.2}$$

where k is a constant still to be defined. Now, the whole two-dimensional space is divided up by the series of circles r_n. For any particular r there are always two nearest circles to be called r_{n-1} and r_n such that

$$r_{n-1} \leqq r < r_n. \tag{9.3}$$

From the definition (9.1) and (9.3) it follows that $\alpha\left(r\right)$ cannot be a continuous function of r as might be suggested by (9.2). Hence, we rewrite (9.2) in a form approaching a linear function as near as possible:

$$\alpha\left(r\right) = \frac{k\left(r_n + r_{n-1}\right)}{2} \quad \text{where } r_{n-1} \leqq r < r_n. \tag{9.4}$$

Substitution in (9.1) yields

$$\left(r_n - r_{n-1}\right)^{-1} = k\left(r_n + r_{n-1}\right), \tag{9.5}$$

or

$$\left(r_n + r_{n-1}\right)\left(r_n - r_{n-1}\right) = \frac{1}{k}, \tag{9.6}$$

or

$$r_n^2 - r_{n-1}^2 = \frac{1}{k}. \tag{9.7}$$

Multiplication by π leads to an unexpected result:

$$\pi r_n^2 - \pi r_{n-1}^2 = \frac{\pi}{k}. \tag{9.8}$$

Substitution of $n = 1$ determines the constant at the right side

$$\pi r_1^2 - \pi r_0^2 = \pi r_1^2 = \frac{\pi}{k}, \tag{9.9}$$

and so

$$\pi r_n^2 - \pi r_{n-1}^2 = \pi r_1^2. \tag{9.10}$$

This recursive formula can be given in words as:

The area of each ring between any two succeeding circles is equal to the area of the innermost circle r_1.

In obeisance to conclusion 4.3 we have to choose the rings alternately coloured and uncoloured. The simplest choice of colour is black which interferes with any colour. In figure 12 the resulting screen trap, the simplest solution of the problem, is given. Interestingly, it is not an uncommon device in optics where it is known as a Fresnel-zone plate.

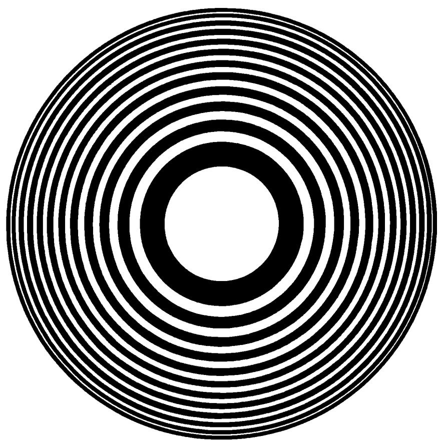

Figure 12. The simplest screen trap which invariably shows a conspicuous moiré pattern with a printing screen of any symmetry in any direction.

The moiré interference pattern with a 4-fold printing screen is shown in figure 13. Note that the trap is repeated manifold. In accordance with conclusion 7.1 it has a 4-fold rotation centre and mirror axes but no translation symmetry. The four innermost images are due to the interference between the principal frequency $\alpha(r)$ of the screen trap and the principal frequency of the printing screen. Further outlying images are due to subharmonics of the screen trap and to subharmonics of the printing screen (compare conclusion 5.2). The contrast of the images diminishes with increasing r. It is easy to deduct that the distance between the centre r_0 and one of the images is a linear function of the frequency of the printing screen.

Figure 13. The moiré pattern of the pattern of figure 12 with a 4-fold printing screen.

It may be well to parenthesize that this screen trap provides more protection than was required in the formulation of the problem at the beginning of this chapter. Because it has a circle symmetry it will interfere not only with a 4-fold printing screen, but also with screens of any other symmetry, surely except 1-fold symmetry.

10 Further demonstrations

A Dutch artist well-known internationally for the great role symmetry played in his drawings was the late Mr. M.C. Escher [8]. Although he did not receive a formal mathematical training as he himself always pointed out, he did incorporate almost all existing two- and three-dimensional groups into his drawings. In 1950 he was invited to submit sketches for a new series of Dutch banknotes. It may be regretted now that his ideas were deemed technically infeasible at the time. As an example one of his designs is reproduced in figure 14 together with its moiré pattern.

(Note in the author's handwriting in the typescript of this paper: *As yet unfinished waiting for print proofs.*)

11 References

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