THE EFFECT OF PAPER FIBRE DIRECTION
ON
THE DURABILITY OF BANKNOTES

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Note

This technical report provides a detailed account of the third circulation trial undertaken since the first generation of banknote sorting machines came into operation at De Nederlandsche Bank in 1975. This digital version was composed in February 2007 and is an edited version of the original text which appeared in 1983.

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Summary

It is an established fact that the paper fibre direction may affect the properties of paper. The circulation trial described in the report at hand aims to quantify the effect of the paper fibre direction on the durability of banknotes. Two fibre directions were examined: parallel with the long side of the banknotes and perpendicular to it. For the purpose of this trial, two practically equal quantities of banknotes differing in the fibre direction were manufactured and brought into circulation. The withdrawal of the notes from circulation was monitored by means of the banknote sorting machines developed by De Nederlandsche Bank, which read the numbers of all banknotes returned. The results were interpreted using the method developed during previous circulation trials, modified to accommodate the Bank’s working stocks of banknotes.

The mean time to failure of 25 notes with the fibre direction parallel to the long side of the notes is 105 weeks, and that of 25 notes with the fibre direction perpendicular to it is 117 weeks. The effect of fibre direction on the life of banknotes in circulation thus appears to be so small that, for practical purposes, no allowance needs to be made for this variable.

The mean time to failure of both types of paper exceeds the mean to failure of 25 notes found in the previous circulation trial considerably. The difference is attributed to unknown factors in the papermaking or printing process.

Keywords
banknotes, failure model, gamma distribution, paper fibre direction, reliability estimation, survival testing.
## Symbols

Capitals are used for quantities describing the entire circulation of $f$ 25 banknotes and small letters for quantities describing test series. As the number of test notes issued may differ per type, the quantities in small letters are expressed as fractions of the numbers originally issued, and are thus dimensionless.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>number of banknotes in circulation of one denomination</td>
</tr>
<tr>
<td>$C_S$</td>
<td>number of fit banknotes in circulation of one denomination</td>
</tr>
<tr>
<td>$C_V$</td>
<td>number of unfit banknotes in circulation of one denomination</td>
</tr>
<tr>
<td>$c$</td>
<td>number of test notes in circulation, fraction of original issue</td>
</tr>
<tr>
<td>$c_s$</td>
<td>number of fit test notes in circulation, fraction of original issue</td>
</tr>
<tr>
<td>$c_v$</td>
<td>number of unfit test notes in circulation, fraction of original issue</td>
</tr>
<tr>
<td>$H$, $h$</td>
<td>hazard rate, ratio of unfit banknotes withdrawn per week to banknotes in circulation</td>
</tr>
<tr>
<td>$i$</td>
<td>cumulative fraction of test notes withdrawn</td>
</tr>
<tr>
<td>$MTIS$</td>
<td>mean time in store</td>
</tr>
<tr>
<td>$MTTF$</td>
<td>mean time to failure</td>
</tr>
<tr>
<td>$O$, $o$</td>
<td>return rate, ratio of banknotes returned per week to banknotes in circulation</td>
</tr>
<tr>
<td>$P$</td>
<td>number of banknotes in stock</td>
</tr>
<tr>
<td>$p$</td>
<td>number of test notes in stock, fraction of original issue</td>
</tr>
<tr>
<td>$Q$, $q$</td>
<td>withdrawal rate, ratio of unfit banknotes withdrawn per week to banknotes returned per week</td>
</tr>
<tr>
<td>$S$</td>
<td>fit banknotes in weekly return</td>
</tr>
<tr>
<td>$s$</td>
<td>fit test notes in weekly return, fraction of original issue</td>
</tr>
<tr>
<td>$V$</td>
<td>unfit banknotes in weekly return</td>
</tr>
<tr>
<td>$v$</td>
<td>unfit test notes in weekly return, fraction of original issue</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>scale parameter of the gamma distribution</td>
</tr>
<tr>
<td>$\eta$</td>
<td>shape parameter of the gamma distribution</td>
</tr>
</tbody>
</table>

1 Introduction

In the nineteenth and early twentieth centuries, Netherlands banknotes were printed on hand-made paper. In 1921, our paper mill Van Houtum & Palm at Ugchelen, The Netherlands, put its first papermaking machine into operation, and, following a period of testing, machine-made paper gradually replaced hand-made paper. That innovation gave rise to a new distinction to be drawn between two types of paper. In contrast to hand-made paper, which is more or less isotropic, machine-made paper is anisotropic. The machine direction is clearly discernible under the microscope from the predominant direction of the fibres, which are more or less oriented in the longitudinal direction of the machine during paper manufacture. The anisotropy may show in the macroscopic paper properties, such as the tensile strength and the folding endurance which tend to be higher if measured on a test strip cut parallel to the machine direction than if measured on a test strip cut perpendicular to it.

Machine-made paper is manufactured in the form of a web. The web is cut into sheets, which are then transported to the printing works to be printed. The sheets can be cut so that the length of the banknotes is either parallel with or perpendicular to the machine direction, as illustrated in figure 1. Because the Dutch tend to fold their banknotes in four with the first sharp fold parallel to the short side of the notes, the Bank of old considered it advisable to print the notes in such a way that the presumably strong machine direction of the paper is parallel to the long sides of the notes. Such notes are referred to as long-grain notes, whereas notes whose long sides are perpendicular to the machine direction of the paper are referred to as short-grain notes.

Figure 1. Long-grain banknotes (left) and short-grain banknotes (right). The paper machine direction is the predominant direction of orientation of the cotton fibres and determines the strongest direction of the paper.
Banknotes are traditionally printed on sheet-fed rotary presses. Recently, web-fed rotary presses — or web presses — have become available to banknote printers. If such web presses are employed, printing the notes perpendicular to the web direction offers a number of advantages in terms of simplicity and efficiency. As explained above, this direction of printing is at right angles to the usual direction for Netherlands banknotes and would, by our assumption, reduce the durability of the banknotes. However, we realised that it was a naive assumption. No experiment had ever been conducted to quantify the effect of the machine direction on the durability of banknotes. As is known, the Netherlands Bank is able to set up and carry out circulation trials successfully by virtue of its sorting machines [5][6]. Since a circulation trial is an excellent instrument to measure the durability of banknotes accurately, we in early 1979 decided to set up a trial with the machine direction of the paper as variable. A batch of paper cut cross-wise was specially manufactured by Van Houtum & Palm and printed by Joh. Enschedé en Zonen at Haarlem, The Netherlands. The resultant short-grain notes were brought into circulation in October 1980, together with an equally large number of long-grain notes to serve as reference. The withdrawal of the notes in the trial, identifiable by their numbers, was monitored by means of the Bank’s sorting machines which record the numbers of all notes returned. This report presents the results.

2 Recapitulation of previous circulation trials

The results of the previous circulation trials are best visualised with the aid of the Venn diagram shown in figure 2. In this diagram, the total circulation of banknotes of one denomination is represented by the set $C$. The circulation $C$ expands gradually over time as a result of such factors as inflation and the growth of the population. It can be divided into two subsets: the unfit, worn, soiled, torn notes $C_V$, which should have been withdrawn already, and the fit notes $C_S$, which may remain in circulation for the time being. Each of these two subsets is more or less constant in proportion to the total circulation; that is the very purpose of the Bank’s sorting system.

Daily, a small portion of the notes in circulation is returned to the Bank. This portion also consists of a fraction of unfit notes, to be called $V$, and a fraction of fit notes, to be called $S$. The previous circulation trials showed that the notes returned represent a random sample from the circulation. On average, the ratio of fit to unfit notes in the sample is then equal to that in the circulation:

$$\frac{C_V (t - 1)}{C (t - 1)} \frac{V (t)}{V (t) + S (t)} \quad \text{and} \quad \frac{C_S (t - 1)}{C (t - 1)} = \frac{S (t)}{V (t) + S (t)}.$$  \hspace{1cm} (1)

The series of test notes which are monitored in a circulation trial constitute another subset $c$ of the circulation. Owing to the withdrawal of unfit
Figure 2. Venn diagram representing the banknote circulation. The unfit and fit banknotes, like the test notes in a circulation trial, constitute subsets of the total circulation.

Notes, this subset of test notes $c$ decreases continually, contrary to the circulation $C$, which expands slowly. As the notes returned daily constitute a random sample from the circulation, the following holds good:

$$\frac{V(t) + S(t)}{C(t-1)} = \frac{v(t) + s(t)}{c(t-1)},$$

where $v$ represents the unfit test notes returned and $s$ the fit test notes returned. To permit comparison of different trials, it is helpful to express $v$, $s$ and $c$ as fractions of the original issue. Further, similar to (1), for the subset of test notes

$$\frac{c_v(t-1)}{c(t-1)} = \frac{v(t)}{v(t) + s(t)} \quad \text{and} \quad \frac{c_s(t-1)}{c(t-1)} = \frac{s(t)}{v(t) + s(t)},$$

where $c_v$ is the subset of unfit test notes in circulation and $c_s$ the subset of fit test notes in circulation. However, it should be noted that

$$\frac{V(t)}{V(t) + S(t)} \neq \frac{v(t)}{v(t) + s(t)}.$$

Koeze [6] proposes a highly simplified model of the circulation of one banknote. The model is based on a Poisson or point process. He conceives that a banknote is subject to a series of elementary events which lead to
wear and tear, $\lambda$ denoting the frequency of these elementary events and $\eta$ their number necessary to make a note unfit. In this very simple model, the withdrawal curve of the notes, the cumulative number of notes withdrawn $i(t)$ as a function of circulation time, can be described by a two-parameter gamma distribution.

Cramer [2][3] describes a similar process but does so from a different angle. He conceives that a note passes along a chain of payments or ‘loop’ through circulation. A loop starts and ends at the counters of a bank. The back of the loop passes through our banknote sorting machines. One of the factors governing the number of loops which a note is capable of passing through is the note quality.

Den Butter and Coenen [4] and Cramer [1] take this further and try to estimate the moment during the last loop through circulation at which a note becomes unfit. In figure 2 that is the moment at which the last loop crosses the fit/unfit boundary line. Generally, this will not coincide with the moment at which the note is returned to the counter of a bank. Unfit notes will usually continue to circulate for some time before being withdrawn.

Table 1, which shows a number of key figures for the Netherlands banknote circulation, presents an attempt at dimensioning the various quantities in the complementary models. In addition to data from the publications referred to above and from the present circulation trial, it also includes figures from the quarterly reports of the banknote operations sector of the Netherlands Bank. All figures relate to the f 25 denomination, printed on 100% cotton banknote paper manufactured by Van Houtum & Palm cut into long-grain notes. The similarities and differences are worthy of note.

The quantities listed in table 1 are the following. The return rate is defined as the ratio of the number of banknotes returned during a week to the number of banknotes in circulation at the beginning of that week:

$$O(t) = \frac{V(t) + S(t)}{C(t-1)}$$

According to (2), $O(t) = o(t)$ on average. The hazard rate is defined as the ratio of the number of unfit banknotes returned during a week to the number of banknotes in circulation at the beginning of that week:

$$H(t) = \frac{V(t)}{C(t-1)}$$

The withdrawal rate is defined as the ratio of the number of unfit notes returned during a week to the total number of banknotes returned during that week:

$$Q(t) = \frac{V(t)}{V(t) + S(t)}$$

(5)
Table 1. Key figures of the f 25 circulation

<table>
<thead>
<tr>
<th>Author</th>
<th>Den Butter &amp; Coenen</th>
<th>Koeze</th>
<th>Koeze &amp; van Gelder</th>
<th>Bank</th>
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<tbody>
<tr>
<td><strong>Return rate</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>total circulation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O mean</td>
<td>0.0597</td>
<td>0.0679</td>
<td>0.0671</td>
<td></td>
</tr>
<tr>
<td>O standard deviation</td>
<td>0.0245</td>
<td>0.0206</td>
<td></td>
<td></td>
</tr>
<tr>
<td>O skewness</td>
<td>0.1445</td>
<td>0.1024</td>
<td></td>
<td></td>
</tr>
<tr>
<td>O kurtosis</td>
<td>3.8558</td>
<td>2.5070</td>
<td></td>
<td></td>
</tr>
<tr>
<td>trial series</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>o mean</td>
<td>0.031 – 0.038</td>
<td>0.0437</td>
<td>0.0708</td>
<td></td>
</tr>
<tr>
<td>o standard deviation</td>
<td>0.0219</td>
<td>0.0265</td>
<td></td>
<td></td>
</tr>
<tr>
<td>o skewness</td>
<td>0.7516</td>
<td>0.1997</td>
<td></td>
<td></td>
</tr>
<tr>
<td>o kurtosis</td>
<td>4.0925</td>
<td>3.0067</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Withdrawal rate</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>total circulation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q mean</td>
<td>0.28</td>
<td>0.26</td>
<td>0.27</td>
<td></td>
</tr>
<tr>
<td>trial series</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q asymptote</td>
<td>0.51</td>
<td>0.33</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Mean time to failure</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>total circulation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MTTF = $H^{-1}$</td>
<td>59.9</td>
<td>56.6</td>
<td>54.2</td>
<td></td>
</tr>
<tr>
<td>trial series</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MTTF = $\eta/\lambda$</td>
<td>47 – 63</td>
<td>54.5</td>
<td>105</td>
<td></td>
</tr>
<tr>
<td><strong>Mean number of loops</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>total circulation</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H^{-1} \cdot O$</td>
<td>3.6</td>
<td>3.8</td>
<td>3.6</td>
<td></td>
</tr>
<tr>
<td>trial series</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\eta/\lambda \cdot O$</td>
<td>3.3</td>
<td>7.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\eta/\lambda \cdot o$</td>
<td>1.5 – 2.4</td>
<td>2.4</td>
<td>7.4</td>
<td></td>
</tr>
<tr>
<td><strong>Number of events</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>trial series</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\eta$ sum total during life</td>
<td>1.27 – 1.45</td>
<td>1.9</td>
<td>2.2</td>
<td></td>
</tr>
<tr>
<td>$\lambda$ average per week</td>
<td>0.023 – 0.027</td>
<td>0.035</td>
<td>0.021</td>
<td></td>
</tr>
<tr>
<td>$\lambda/O$ average per loop</td>
<td>0.6 – 0.9</td>
<td>0.6 – 0.8</td>
<td>0.3</td>
<td></td>
</tr>
</tbody>
</table>

Cramer [2] estimates the average number of events per loop at 1.5.
These quantities are interdependent, of course:

\[ H(t) = O(t) \cdot Q(t) \quad \text{and} \quad h(t) = o(t) \cdot q(t). \]  

(8)

The mean time to failure \( MTTF \) can be calculated in various ways. The Banknote Printers’ Conference (BPC) uses a nonparametric formula, viz. the ratio of the average circulation during a year to the total number of notes withdrawn during that year. This is the reciprocal of the hazard rate taken over one year:

\[ MTTF = \frac{C(t-1)}{V(t)} = H^{-1}(t). \]  

(9)

The period of one year is more or less arbitrary, but it does exclude any seasonal effects. The nonparametric BPC formula permits the calculation of \( MTTF \) for the circulation of one denomination, or several, or all denominations.

The mean time to failure \( MTTF \) can also be calculated if it is assumed, on more or less plausible grounds, that the withdrawal of banknotes satisfies one of the usual statistical distribution functions. They have the advantage that \( MTTF \) can be calculated separately for any subset of the circulation. In the case of the gamma distribution with parameters \( \eta \) and \( \lambda \),

\[ MTTF = \frac{\eta}{\lambda}. \]  

(11)

where \( \eta \) is the total number of elementary events necessary to make a note unfit and \( \lambda \) the frequency of these events. Both the nonparametric and the parametric method assume that the total circulation is stationary.

The average number of loops is the mean time to failure \( MTTF \) multiplied by the return rate \( O \) or \( o \). In principle, both the nonparametric and the parametric methods can be used for this purpose. In the former case, the calculation is identical with the reciprocal of the withdrawal rate \( Q \):

\[ MTTF \cdot O(t) = \frac{V(t) + S(t)}{V(t)} = Q^{-1}(t). \]  

(12)

The average number of loops is, of course, equal to the average number of times that the notes are returned to the Bank. This includes the last time, when the notes are sorted out as unfit and destroyed.

The average number of events per loop is the average number of events divided by the average number of loops during the life of the banknotes. As there is no precise definition of ‘event’, the various estimates do not necessarily concern the same quantity. The differences between the estimates may be
due to differences in definitions and, hence, in estimation methods. Economists may be tempted to define a loop as running between two counters of the Post Office or the commercial banks, and an event as a real economic payment. From an engineering point of view it may be more meaningful to define a loop as running between two branch offices of the Central Bank because every physical transference involves the risk of wear to the banknotes and is thus a potential event.

3 The effect of notes held in stock at the Bank

In the two circulation trials conducted earlier [5] and [6] we found that the return rate \( o(t) \) of the test series was substantially lower than the return rate \( O(t) \) of the total circulation of the denomination concerned. However, according to (2) and (5), the two rates should be equal. On consideration we thought it likely that the difference might be due to the notes held in stock at the Bank. Consequently, as from the start of this third trial, the stocks of notes have been recorded.

3.1 Circulation

The Bank’s stocks comprise a large and varying number of banknotes. On the one hand, the notes returned from circulation are continually added. On the other hand, the notes approved for recirculation are reissued and the unfit notes are destroyed. Four categories of notes in stock can thus be distinguished:

— circulated unsorted notes,
— circulated sorted fit notes,
— circulated sorted unfit notes to be destroyed, and
— uncirculated new notes.

The fourth category is not relevant to a circulation trial. As all series in the trial are issued in a brief period, this category does not contain any test notes. The third category is likewise irrelevant for our purposes, as after sorting the unfit test notes are immediately added to the cumulative fraction of withdrawn notes \( i \). The first two categories are, however, relevant as they will generally contain test notes. Below, references to ‘the stocks of notes’ or ‘notes in stock’ will hence relate to the stocks of the first two categories.

Although the size of the stocks is, from moment to moment, precisely known, the quantities of test notes in stock, fit and unfit, yet remain a guess. It is true that, as the sorting machines at the Bank record the numbers of
all the notes which pass through the machines by date, it is possible to trace the moment a single note is returned and reissued. However, in practice, it is impracticable to do so for large quantities of notes. As part of the daily routine, the quantities of test notes contained in the stocks can only be estimated.

In the previous circulation trials [5] and [6] we found the probability of being returned to the Bank to be equal for all banknotes in circulation. In other words, the notes in stock constitute a random sample from the total of notes in circulation. The Bank’s stocks will thus contain the same proportion of test notes as the total circulation. If \( P \) represents the total number of notes in stock, \( p \) the number of test notes in stock, \( C \) the total number of notes in circulation, and \( c \) the number of test notes in circulation, as a direct consequence of (2) the following holds good in the stationary situation:

\[
\frac{P(t)}{C(t-1)} = \frac{p(t)}{c(t-1)}.
\]  

(13)

Further, a test note is either in circulation, or in stock, or withdrawn so that

\[
c(t) + p(t) + i(t) = 1,
\]  

(14)

where \( i(t) \) is the cumulative number of withdrawn test notes. Like \( c \), we express \( p \) and \( i \) as fractions of the original issue. If we may approximate (13) by

\[
\frac{P(t)}{C(t)} = \frac{p(t)}{c(t)}.
\]  

(15)

after elimination of the unknown \( p \) it follows for the test notes in circulation

\[
c(t) = \{1 - i(t)\} \frac{C(t)}{C(t) + P(t)}.
\]  

(16)

The quantities \( i \), \( C \) and \( P \) all are accurately known at the Bank at any moment. This equation supersedes our equation (11) in [6].

Equation (15) is a first-order approximation, because, though \( C \) is more or less constant, \( c \) certainly is not as the number of test notes in circulation decreases continually. The result is a phase shift between \( c \) and \( p \). However, if the decrease of \( c \) is not too rapid, (15), and thus (16) also, represents an adequate approximation. The accuracy of (16) is hence limited by two causes: statistical variations and the nonstationary nature of the circulation of test notes.

The new calculation of \( c \) allows the return rate \( o \) and the hazard rate \( h \) to be computed in the usual manner by (5) and (6). Naturally, as thus far the number of test notes in circulation has been overestimated, both quantities will come out higher than before. We shall show in subsection 5.1 that the anomalous differences in return rates between the test series and the total circulation, which we observed in [5] and [6] for the first time, are greatly reduced as a result. The differences between \( O \) and \( o \) remaining after correction for the stocks, given in table 1, are nonsignificant.
3.2 Swinging-in phenomenon in return rates

For small values of $t$ during the first few weeks after issue of the test series, \[15\] will not be valid as it takes some time before the stocks have turned over. Due to the size of the stocks of notes and to the limited handling capacity of our transport and sorting system, there is a certain time lag before a stationary equilibrium is established, during which exceptional behaviour can be expected of the return rate. On the basis of the known size of the stocks and the known handling capacity of the transport and sorting systems, the characteristic time constant can be estimated without undue difficulty. In subsection \[5.1\] we shall demonstrate that, if the adjustment of equation \[16\] is applied, the return rates $o$ exhibit an oscillatory behaviour of second- or higher-order nature with the expected time constant.

3.3 Mean time in store

A third effect of the stocks of notes is more difficult to incorporate satisfactorily into the model. The effect itself is plain enough: if notes are in stock, they do not age in circulation. The stocks seemingly prolong the life of the banknotes. As measurement of the time in store of each individual note is not practicable, this effect can only be modelled in an approximate manner. In a stationary situation, if $S$ notes are returned per unit of time and the same quantity is issued so that stocks $P$ are constant, the time in store of a note is $P/S$. We therefore approximate the time in store at moment $t$ by

$$\frac{P(t)}{S(t)}.$$ \hspace{1cm} (17)

In section \[2\] we argued that a test note is returned to the Bank $MTTF \cdot o$ times. This includes the last time, when the note is withdrawn as being unfit. This last time should not be included in the calculation of the mean time in store, because, upon sorting, an unfit test note is immediately added to the cumulative fraction of withdrawn notes. Consequently, the number of times which a note passes through the stocks as fit for circulation is $MTTF \cdot o - 1$. If $MTIS$ represents the mean time in store, it then follows that

$$MTIS = (MTTF \cdot o - 1) \left\{ \frac{1}{t} \sum_{u=1}^{t} \frac{P(u)}{S(u)} \right\}.$$ \hspace{1cm} (18)

The period on which the mean is calculated is arbitrary. The duration of the circulation trial is an obvious choice.

The mean time in store calculated in accordance with \[18\] must have the same value as the characteristic time constant discussed in section \[3.2\]. So, three methods to quantify $MTIS$ are available: (i) estimation based on the logistic system with known waiting times, (ii) observation of the empirical
time constant of the return rate, and (iii) calculation in accordance with \[ (18) \] on the basis of observed quantities. The three methods should come out close, of course.

3.4 Mean time to return

It is straightforward to estimate the time elapsing between the moment at which a note becomes unfit and its return to the Bank. In our simple Poisson model of \[ \text{[6]} \] the return of unfit notes follows an exponential distribution \[ [1] \]. As the mean number of notes returned per week is \( o \), the mean time to return \( MTTR \) elapsing between the moment at which a note becomes unfit and the moment of its return to the Bank is

\[
MTTR = o^{-1}.
\] (19)

3.5 Actual technical life

If we may assume that the withdrawal of the test notes follows a gamma distribution, the apparent mean time to failure \( MTTF \) of the test notes follows by the method of Wilk et al. \[ 7 \], just as in the previous circulation trials. The discussion in this section enables us now to adjust this apparent mean time to failure \( MTTF \) for the mean time in store \( MTIS \) and for the mean time to return \( MTTR \) to obtain the actual technical life \( MTTF^* \) of the test notes:

\[
MTTF^* = MTTF - MTIS - MTTR.
\] (20)

4 Implementation of the circulation trial

The two earlier circulation trials described in \[ 5 \] and \[ 6 \] showed that such trials are well suited to compare the durability of different types of experimental banknotes. A number of about 600,000 notes of each type is amply sufficient to permit the collection of accurate and statistically reliable data. However, doubts arose as to the reproducibility of the results. The above and other circulation trials described by Den Butter and Coenen \[ 4 \], which were carried out in different periods, yielded widely differing results as regards the mean time to failure of the notes, even if the paper qualities were nominally identical. One possible cause could be the fact that the batches of paper had been manufactured and printed at different points of time, giving rise to unintended differences among the batches, for instance, due to ageing of the paper. Moreover, batches of the size mentioned are relatively minor in terms of the production capacity of a paper mill or printing works: a day’s output or even less. Consequently, slow nonreproducible variations in production could be another factor accounting for disturbing differences among
the paper types. To eliminate these two potential causes of differences between the short-grain paper to be used in this trial and the long-grain paper to serve as the reference, the two batches of paper were manufactured and printed one immediately after the other. This had not been done in the two earlier circulation trials, when an arbitrary batch of new notes in stock was taken as the reference.

The printers supplied $1.1 \times 10^6$ short-grain notes and $1 \times 10^6$ long-grain notes. The denomination chosen was $f 25$. Of each type of paper, six series were selected for the circulation trial, as detailed in table 2. The usual laboratory tests, the results of which are given in table 3 showed the two batches to meet our banknote paper specifications.

Table 2. Details of the test series.

<table>
<thead>
<tr>
<th>Paper type</th>
<th>number of notes issued</th>
<th>fraction withdrawn at 28 January 1983</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long-grain</td>
<td>585,100</td>
<td>0.6916</td>
</tr>
<tr>
<td>Short-grain</td>
<td>590,900</td>
<td>0.6560</td>
</tr>
</tbody>
</table>

Table 3. Results of the laboratory tests.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>circumstances</th>
<th>unit</th>
<th>long-grain</th>
<th>short-grain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>g/ m²</td>
<td>81</td>
<td>82</td>
<td></td>
</tr>
<tr>
<td>Moisture expansion</td>
<td>cross direction,</td>
<td>%</td>
<td>2.77</td>
<td>2.54</td>
</tr>
<tr>
<td></td>
<td>1/2 hour in water</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Off colour</td>
<td>judd</td>
<td>0.8</td>
<td>1.6</td>
<td></td>
</tr>
<tr>
<td>pH</td>
<td>cold extraction</td>
<td>5.5</td>
<td>5.7</td>
<td></td>
</tr>
<tr>
<td>Opacity</td>
<td>%</td>
<td>86.5</td>
<td>86.3</td>
<td></td>
</tr>
<tr>
<td>Printing opacity</td>
<td>%</td>
<td>4.5</td>
<td>4.1</td>
<td></td>
</tr>
<tr>
<td>Tensile strength</td>
<td>cross direction</td>
<td>kN/m</td>
<td>3.8</td>
<td>3.8</td>
</tr>
<tr>
<td>Wet strength retention</td>
<td>cross direction,</td>
<td>%</td>
<td>37.8</td>
<td>32.1</td>
</tr>
<tr>
<td></td>
<td>1 hour in water</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Folding endurance</td>
<td>machine direction</td>
<td></td>
<td>2105</td>
<td>1810</td>
</tr>
<tr>
<td>Folding endurance</td>
<td>cross direction</td>
<td></td>
<td>921</td>
<td>1114</td>
</tr>
<tr>
<td>Evenness Bekk</td>
<td>top side</td>
<td>s</td>
<td>42</td>
<td>28</td>
</tr>
<tr>
<td>Evenness Bekk</td>
<td>wire side</td>
<td>s</td>
<td>43</td>
<td>37</td>
</tr>
<tr>
<td>Air permeability Bekk</td>
<td>s</td>
<td></td>
<td>83</td>
<td>109</td>
</tr>
</tbody>
</table>

The notes were issued through the Bank’s Head Office in Amsterdam in the period Monday 6 October 1980 until Friday 17 October 1980. For the evaluation, Friday 10 October 1980 is taken as the date of issue. The notes from the test series were handled in the usual manner, that is they were sorted on the banknote sorting machines mixed with the other notes.
returned from circulation. As from the date of issue, records were kept of the notes sorted. On Friday 28 January 1983, the last day covered by the trial, the notes had been in circulation for 120 weeks and about 70% of each type had been withdrawn.

5 Results

5.1 Return rate

The return rates $O$ and $o$ for the entire $f\ 25$ circulation and the test notes, respectively, are defined in formula (5), where the circulation $c$ of the test notes is given by equation (16). The prime importance of the return rate in a circulation trial is that it permits the proper performance of the trial to be monitored, as it is highly sensitive to operational errors which may occur during note sorting. If it can be established that the expected value of the return rate was constant throughout a trial and equal for all banknote series in the trial and the circulation as a whole, the further results, such as those having to do with the durability, are reliable.

In the present circulation trial, the usual statistical properties of the return rates were checked. The correlation coefficient between the return rates of the long-grain and the short-grain paper again proved to be very high: +0.9988 during the 120 weeks of the trial. The correlation coefficients between the entire $f\ 25$ circulation and the test series are

long-grain paper $r = +0.7632$;
short-grain paper $r = +0.7584$.

The high degree of correspondence between the return rates is also evident from figure 3 in which the return rates of the two paper types have been plotted as functions of the circulation time since issue. For comparison, figure 4 shows the return rate of the entire $f\ 25$ circulation.

If, using the least squares method, straight lines are fitted to the data sets for the two types of paper and if the t-statistics are calculated under the hypothesis that the slope of the lines is zero, these are not significant at the 0.05 significance level for $t \leq 93$ weeks. Under the hypothesis that the two data sets are independent, normally distributed random variables with equal means, at $t = 93$ weeks the t-statistic appears to be +0.0752. Hence, we may infer that the return rates $o$ are constant and equal during the first 93 weeks. The moment $t = 93$ we regard as the time limit beyond which the results become unreliable. After that moment, the return rates fall off as a result of the cullings and rejects which happen during sorting of the notes, as we showed already in [6]. All further results we base on the first 93 weeks.

The mean return rates $\overline{o}$ of the test series are
Figure 3. Return rates of long-grain and short-grain paper.
\( o = \) long-grain paper, \( + = \) short-grain paper, \( * = \) coincident dots.

Figure 4. Return rate of the entire f 25 circulation.
long-grain paper $\bar{r} = 0.0706 \text{ week}^{-1}$ ($-1.9349$),
short-grain paper $\bar{r} = 0.0709 \text{ week}^{-1}$ ($-1.9307$),
where the figures in parentheses are the t-values if the slopes are zero, while the mean return rate $\bar{O}$ of the entire $f \ 25$ circulation is

all $f \ 25$ series $\bar{O} = 0.0679 \text{ week}^{-1}$.

Under the hypothesis that the latter is equal to the means of the test series, the t-statistic is $-0.7967$ for long-grain paper and $-0.8785$ for short-grain paper. As these two values are not significant at the 0.05 significance level, the hypothesis that the return rates of the test series and the entire $f \ 25$ circulation are equal is not rejected. The conclusion is evident. We may be satisfied not only that the mean return rates of the two types of paper are constant and equal to each other but also that they are equal to the mean return rate of the $f \ 25$ circulation as a whole.

Figure 5. Difference between return rates for long-grain paper and the entire $f \ 25$ circulation. Note that during the first 12 weeks the return rate shows an oscillatory phenomenon of a higher order and that the return rate of long-grain paper decreases gradually with increasing circulation time.

Figure 5 shows the difference $o - O$ between the return rates for long-grain paper and the entire $f \ 25$ circulation. The difference $o - O$ between the return rates for short-grain paper and the entire $f \ 25$ circulation shows practically the same pattern. During the first 10 weeks the difference shows an oscillatory phenomenon of a higher-order nature, peaking between $t = 4$ weeks and $t = 7$ weeks. This suggests a system of at least second order with a time constant of about 6 weeks. The observed time constant corresponds
to the time constant estimated on the basis of the handling capacity of our transport and sorting systems, which also proves to be about 6 weeks, as noted in subsection 4.2. After some 12 weeks the stocks have been turned over completely.

The pattern presented in the figure does not appear to be a pure, undisturbed oscillatory phenomenon; this might be due to the fact that the notes were issued over a period of two weeks rather than a single day. For a mathematical description of the observed pattern, an undisturbed oscillation would have to be convoluted with the issue spread over two weeks. That, however, we consider to be beyond the scope of this paper.

Finally, figure 5 again shows that the return rate of long-grain paper, and the short-grain paper for that matter, falls off gradually, ultimately dropping below that of the entire f 25 circulation.

5.2 Withdrawal rate

The quantities having the greatest operational significance are the withdrawal rates $Q$ and $q$ as defined in formula (7), describing as they do the fraction of notes withdrawn in the notes returned from circulation. An increasing withdrawal rate drives up the costs in new notes to keep the circulation clean. Figure 6 presents the withdrawal rates $q$ for long-grain and short-grain paper, while figure 7 shows the withdrawal rate $Q$ for the entire f 25 circulation.

It is noteworthy that both paper types take much longer to wear than the f 25 notes in the preceding circulation trial described in [6]. The asymptote of the withdrawal rate in the present trial is 0.3 after 70 weeks. In the preceding trial the asymptote was 0.5 which was reached after 25 weeks already. This is a considerable difference indeed. The sorting process operated identically: in each of the two trials a fraction of about 0.27 of all f 25 notes returned was withdrawn. Consequently, the difference between the two asymptotes must be due to quality of the test notes. There must be factors in banknote manufacturing, the production of the paper or the printing of the notes, which cause major differences in the life of the notes and are yet unknown and not under control.
Figure 6. Withdrawal rates of long-grain and short-grain paper.

- o = long-grain paper,
- + = short-grain paper,
- * = coincident dots.

Figure 7. Withdrawal rate of the entire f 25 circulation.
5.3 Withdrawal curve

The withdrawal curve $i(t)$ is defined in (10) as the cumulative fraction of notes sorted out as unfit and, hence, withdrawn. Out of all experimental quantities $i(t)$, being an integral which smoothes statistical irregularities, is the best variable to quantify the durability of the notes accurately. If $i(t)$ satisfies one of the usual statistical distribution functions, it is possible to estimate the mean time to failure from the observed parameters. In the previous circulation trials [5], [6] and [4] it was shown by experiment and reasoning that the gamma distribution provides an adequate statistical description.

Figure 8. Withdrawal curves of long-grain and short-grain paper.

$\circ =$ long-grain paper, $+$ = short-grain paper, $*$ = coincident dots.

However, the withdrawal curves found in the present trial do not allow of a simple explanation. The withdrawal curves of the two paper types, shown in figure 8, appear to intersect each other after 50 weeks of circulation. From figure 6, too, it is evident that during the first 30 weeks the long-grain paper stayed in circulation longer than the short-grain paper. This is in conformity with the naive assumption set forth in section 1. However, for circulation times in excess of 30 weeks the situation was reversed in that the long-grain paper was withdrawn more rapidly than the short-grain paper. Not only is this in contrast to the naive assumption, but neither is it easy to provide an easy explanation. Possibly the withdrawal curves are of a composite nature. During the initial period, in conformity with the naive assumption, the occurrence of sudden catastrophic failures, such as tears, might be the main cause to withdraw notes from circulation, whereas during the later period degradation failures, such as soiling, might take over. However, from the observations in this trial we have no clue as to what happened. A solution would be to record the withdrawal curve for each failure mode separately.
but, unfortunately, we have not done so.

In principle, the usual statistical distributions cannot account for intersecting withdrawal curves. For want of a better solution, we have yet calculated the mean time to failure with the aid of the gamma distribution. Using the method of Wilk et al. [7], the maximum likelihood parameters of the gamma distribution are estimated for the two paper types at

long-grain paper \( \lambda = 0.0208 \text{ week}^{-1}, \quad \eta = 2.19 \),
short-grain paper \( \lambda = 0.0155 \text{ week}^{-1}, \quad \eta = 1.81 \).

It follows that the mean times to failure are

long-grain paper \( MTTF = 105 \text{ weeks} \),
short-grain paper \( MTTF = 117 \text{ weeks} \).

The difference between these two values is so small that for practical purposes the paper machine direction may be left out of account.

It is, however, most remarkable that both values are much larger than the mean time to failure found so far for \( f \) 25 notes. The life of \( f \) 25 notes found during previous trials and the average life for the entire \( f \) 25 circulation are both some 55 weeks. Table I summarises the various results. Even though all conceivable precautions were taken in this trial to avoid accidental differences in life, as described in section 4 above, the mean time to failure proved to be much longer. Our conclusion is, yet again, that during banknote manufacturing major factors are operative which are yet unknown and not under control.

As this was the first circulation trial in which we make allowance for stocks of notes, we are now able for the first time to make a rough adjustment for the effects of the stocks on the life of notes. During the first 93 weeks, the time in store according to (17) was 8.4649 weeks on average after each return to the Bank. According to (18) the mean time in store \( MTIS \) was then

long-grain paper \( MTIS = 54.5 \text{ weeks} \),
short-grain paper \( MTIS = 61.6 \text{ weeks} \).

Because the mean time to return \( MTTR \) is equal to \( o \text{ week}^{-1} \) according to (19), it follows that the actual technical life \( MTTF^* \) is by (20)

long-grain paper \( MTTF^* = 36.7 \text{ weeks} \),
short-grain paper \( MTTF^* = 41.1 \text{ weeks} \).

Using the figures from the quarterly reports of the banknote operations sector of the Bank (see table I last column) the actual technical life \( MTTF^* \) of the entire \( f \) 25 circulation is

\[
MTTF^* = 54.2 - (3.6 - 1) \cdot 8.4649 - (0.0671)^{-1} = 17.3 \text{ weeks}.
\]

The accuracy of \( MTTF^* \) is yet unproven; the figures must therefore be used with caution.
6 Conclusions

1. The effect of paper fibre direction on the life of banknotes in circulation is so minor that, for practical purposes, no allowance needs to be made for this variable.

2. The previously published numerical model of wear and tear of banknotes in circulation has been extended with a simple and practical correction for the notes in stock at the Bank. The correction increases the return rate of the test series so that the differences with the return rate of the entire circulation which were previously observed are reduced to nonsignificant ones.

3. The time the notes spend in the Bank’s stocks can also account for an oscillation of the return rate of the test series during the first 10 weeks of a circulation trial.

4. The mean time to failure of a particular batch of banknotes can be twice as long as the mean time to failure averaged over the circulation as a whole. The factors which cause the longer life are not yet known.

5. The actual technical life of banknotes has been estimated for the first time. The apparent mean time to failure has been corrected for the time notes spend in the Bank’s stocks and the time they need to return to the Bank once they become unfit during circulation.
References


